THE COSMOLOGICAL MASS DISTRIBUTION

FROM CAYLEY TREES WITH DISORDER

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ABSTRACT. We present a new approach to the statistics of the cosmic density field and to the mass distribution of high-contrast structures, based on the formalism of Cayley trees. Our approach includes in one random process both fluctuations and interactions of the density perturbations. We connect tree-related quantities, like the partition function or its generating function, to the mass distribution. The Press & Schechter mass function and the Smoluchowski kinetic equation are naturally recovered as two limiting cases corresponding to independent Gaussian fluctuations, and to aggregation of high-contrast condensations, respectively. Numerical realizations of the complete random process on the tree yield an excess of large-mass objects relative to the Press & Schechter function. When interactions are fully effective, a power-law distribution with logarithmic slope -2 is generated.

Subject headings: cosmology: theory – galaxies: clustering – large scale structure of the universe – galaxies: formation

1. INTRODUCTION

The mass distribution $N(M)$ of high-contrast structures constitutes a central link between observed extragalactic sources and the physics of the early universe.

In the canonical scenario such structures form by direct hierarchical collapses (DHCs). In the expanding Universe small overdensities above the decreasing background density ρ are weakly gravitationally unstable (see Peebles 1993). The contrasts $\delta \equiv \Delta\rho/\rho$ grow slowly in their linear regime, with $\delta \propto t^{2/3}$ in a critical

universe. As the contrasts approach unity the perturbations detach from the Hubble expansion, turn around, and in a comparable time non linearly collapse to end up in high-contrast virialized structures.

As for the "initial" conditions, the physics of the early universe (see Kolb & Turner 1990) suggests the perturbations started in the form of a Gaussian random field. Their power spectrum at $z < 10^3$ may be piecewise approximated by $|\delta_k|^2 \propto k^n$ with $-3 < n < 1$. So nonlinear conditions $k^3|\delta_k|^2 \sim 1$ are reached sequentially at larger and larger sizes, with the rich clusters (Abell 1958) forming now.

For the range of masses $M \gtrsim 10^{12} M_\odot$ the collapses are little affected by dissipation, and the DHC scenario gives rise yo a theory of elaborate elegance (see Peebles 1965; Gunn & Gott 1972; Press & Schechter 1974, hereafter PS; Rees & Ostriker 1977; Bond et al. 1991). The *linear* density field δ is smoothed or averaged at each point with a filter function of effective size R corresponding to a mass $M \propto \rho R^3$. On each scale M, the variance of $\delta(M)$ yields the dispersion $\sigma \propto M^{-a}$ ($a \equiv 1/2 + n/6$) of the "initial" Gaussian distribution $p(\delta, \sigma)$. The *nonlinear* collapses are modeled after the pattern provided by the top-hat smoothing filter; namely, isolated spherical and homogeneous overdensities which virialize, with a definite characteristic mass $M_c(t) \propto t^{2/3a}$ in a critical universe, by the time when the actual contrasts reach values $\sim 2\ 10^2$. As this time would correspond to a nominal contrast $\delta_c \approx 1.7$ in the extrapolated linear behavior, such threshold is taken to separate the linear regime from bona fide condensations.

The condensed mass fraction is provided by the fraction of spheres where the linear $\delta(M)$ crosses the threshold δ_c. The mass assigned to each collapsed object is twice that of the largest sphere wherein such a condition applies, while the smaller ones are disregarded. This elaborate selection rule can be written simply as $N(M) \, M \, dM = -2\rho \, d \int_{\delta_c}^{\infty} d\delta p(\delta, \sigma)$, as originally proposed by PS, and in terms of $m \equiv M/M_c(t)$ and $\sigma \propto m^{-a}$ yields

$$N(M,t) = \rho \, \frac{2a\delta_c}{\sqrt{2\pi}M_c^2(t)\sigma} \, m^{-2} e^{-\delta_c^2/2\,\sigma^2} \,. \tag{1.1}$$

The single parameter δ_c should comprise the complex nonlinear dynamics of the collapses. Once its value is set on the basis of, e.g., the top-hat model, the functional form of eq. 1.1 at any given t self-similarly depends on the initial spectrum only; specifically, on the spectral index n and on the amplitude taken by σ at the scale $8h^{-1}$ Mpc singled out by unit variance in galaxy counts. Self-similarity is stressed in the analysis of Bond et al. 1991, who examine the linear field at *one* epoch (say, the present t_o) on the resolution $S \propto M^{-2a} \propto \sigma^2$, and compare the resulting $\delta(S)$ with a lowering threshold $1.7 \, (t/t_o)^{-2/3}$.

The same authors clarify in terms of excursion sets the statistics underlying eq. 1.1. They show that $\delta(S)$, when extracted from the Gaussian field using a sharp k-space filter, executes a simple (Markovian) random walk governed by a diffusion equation. The diffusive flux density per unit resolution of such trajectories, having their first up-crossing through the threshold δ_c within dM at M, yields the PS expression complete with shape and amplitude, in agreement with the selection rule.

For all its background, the result 1.1 has a number of drawbacks. The above prescriptions avoid overcounting substructures and imply a uniform timescale $3t/2$ for all collapses. By the same token, however, they are likely to underplay substructures and to overestimate the normalization. In fact, computations which assign definite sizes and timescales to the collapses (Cavaliere, Colafrancesco & Scaramella 1991) yield longer permanence of substructures and a lower normalization for $N(M, t)$. Equivalently, filters localized in the ordinary rather than in the k-space (Bond et al. 1991) change the mass assignment to the collapsing density peaks and yield different statistics and generally straighter shapes. Observations, especially in X-rays, image abundant substructures within clusters, ranging up to truly binary configurations (see Jones & Forman 1992; White, Briel, & Henry 1993).

In addition, a considerable body of evidence indicates that the direct collapse scenario is *incomplete* at the high-mass end. For example, at galactic scales the cD's outnumber the Schechter luminosity distribution (Schechter 1976; Bhavsar 1989); the building up of their bodies is best understood in terms of aggregations of normal members in groups as described by the Smoluchowski kinetic equation (Cavaliere, Colafrancesco, & Menci 1992, hereafter CCM). Galaxy interactions and merging involving one bright partner are also likely to stimulate the emissions from active galactic nuclei at $z \lesssim 2.5$, as indicated by statistics, by morphologies of the host galaxies, and by the richness of their environments (see Heckman 1993; Bahcall & Chokshi 1991); numerical experiments and theoretical studies

also concur to this view (Shlosman 1990; Barnes & Hernquist 1991). At larger scales, the imaging X-ray observations referred to above provide many snapshots of groups and clusters at various stages of essentially binary aggregations.

In fact, high-resolution N-body experiments (e.g., Brainerd & Villumsen 1992; Katz, Quinn, & Gelb 1993; Jain & Bertschinger 1994) show structure formation to be a far more complex process than envisaged by the simple DHC scenario. It includes, in addition to direct collapses, frequent encounters and aggregations within clusters and within larger scale, precursor structures with the dimensionality of sheets and filaments. The resulting $N(M,t)$ shows, relative to eq. 1.1, a slower evolution and a different shape due to an excess at large M.

Here we propose a novel approach to a satisfactory $N(M,t)$. We discuss in §2 and §3 how the mass distribution is generated by a *complete* statistics in the resolution-contrast plane. This comprises as limiting cases both the direct collapses from independent Gaussian fluctuations (§4), and pure dynamical aggregations of high-contrast condensations (§5). The competition and mixing between these two components is computed in §6, with a net outcome depending on the mass range and on ambient conditions. This balance is of keen interest because interactions are likely to contribute, as noted above, key common features of apparently diverse astrophysical phenomena in the nearby and in the distant Universe.

2. DISORDER AND BRANCHING

The language of Cayley trees with disorder (see Derrida & Spohn 1988) con-

veniently describes at the "microscopic" level of density contrasts δ how new collapses from the linear perturbations compete or combine with aggregations of the existing condensations.

The tree is a computational structure (visualized by fig. 1) where at each step μ random weights w_i are extracted in a cascade following a sequence of links, which may randomly branch into two. As the generation number μ increases, the progressive product of such random weights w_i will be related to the probability of finding a fluctuation of the density field. The tree coordinate μ will be related to the mass scale, and the probability will be related to the number of condensations per unit mass. The end result at the "macroscopic" level will be a mass distribution $N(M, t)$.

The tree includes in one random process the following two components of the δ field: (1) disorder - with increasing resolution $S \propto \sigma^2 \propto M^{-2a}$ the independent values taken at a given time by $\delta(M)$ execute, as recalled in §1, a pure *random walk*; and (2) branching - δ may also jump by stochastic "*branching*", actually coalescing two paths into one.

The statistics of the combined process is conveniently derived in terms of the partition function computed at each generation μ along the tree in the direction of coalescence from an initial μ_o: $Z_\mu = \sum_{[paths]} \Pi_{i=\mu_o}^{\mu} w_i$, where the product refers to the i-th preceding serial links of the tree, and the sum includes all paths coalesced at μ. The distribution function of Z will be $P_\mu(Z)$, with moments

$$\langle Z^k \rangle_\mu \equiv \int dZ \, P_\mu(Z) \, Z^k \ . \tag{2.1}$$

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It proves technically convenient to set $w_i = 1 + v_i$, to exhibit the unbiased average $\overline{w}_i = 1$ when v fluctuates around 0. In this representation the partition function reads [1]

$$Z_\mu = \sum_{[paths]} \Pi_{i=\mu_o}^\mu (1 + v_i) \ . \tag{2.2}$$

For the specific tree in fig. 1 the definition of Z implies the following recursion relations to hold at each elementary step $d\mu$:

$$Z_{\mu+d\mu} = \begin{cases} (1 + v) \, Z_\mu & \textit{with probability } 1 - \eta d\mu, \\ (1 + v)[Z_{1\mu} \, u_1 + Z_{2\mu} \, u_2] & \textit{with probability } \eta d\mu. \end{cases} \tag{2.3}$$

The first line describes pure disorder, and the second includes branching; v and u are stochastic variables. Gaussian initial conditions for the perturbation field

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[1]

For $v \to 0$, the limit we shall consider, this is indistinguishable from the other representation $Z_\mu = \sum_{[paths]} e^{-\sum_{i=\mu_o}^\mu v_i}$. The latter is heuristically attractive because one may directly identify $\sum_i v_i$ with $\sum_i \delta_i = \delta$ and use the relationship $\delta = E/E_b$ with the energy E of linear perturbations in a critical universe normalized to the background potential energy (Peebles 1980). The resulting $Z_\mu = \sum_{[paths]} e^{-\sum_{i=\mu_o}^\mu E_i/E_b}$ has the form of a standard partition function, and the counting expressed by the definition (2.2) implies a sum rule for the linear energies. Nonlinear superpositions of comparable energy fluctuations are explicitly accounted for by $\sum_{[paths]}$ in the definition (2.2) and corresponding to the relation (2.3b).

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imply for v a distribution which for small v goes into a Gaussian with variance $D\,d\mu$ proportional to the step length:

$$g(v) = e^{-v^2/2Dd\mu}/(2\pi Dd\mu)^{1/2} \ . \tag{2.4}$$

Another stochastic function $r(u)$, generally far from symmetric, governs the distribution of the weights u translating, as we shall show, the interaction probabilities.

A compact way to embody all moments is in terms of the generating function

$$G_\mu(x) = \langle e^{-(1+x)Z_\mu} \rangle \ . \tag{2.5}$$

Successive derivatives of G at $x = 0$ are related to moments of Z of increasing order, as shown by the formal expansion

$$G_\mu(x) = \sum_k (-1)^k \, (1+x)^k \langle Z^k \rangle_\mu / k! \ \ . \tag{2.6}$$

Correspondingly, the evolution of the moments can be embodied in a master differential equation equivalent to the recursion relations eq. 2.3. It follows from eq. 2.5 that in an interval $d\mu$ the two components simply add with the weights provided by their probabilities as given eqs. 2.3, and their superposition gives

$$G_{\mu+d\mu}(x) = (1 - \eta d\mu)\,\overline{G}_\mu + \eta\,d\mu\tilde{G}_\mu^2 \qquad with \tag{2.7}$$

$$\overline{G}_\mu \equiv \int dv\, g(v)\, G_\mu(x+v+vx), \quad \tilde{G}_\mu \equiv \int du\, r(u)\, G_\mu(x\,u) \ .$$

The continuous limit $d\mu \to 0$ yields

$$\partial_\mu G_\mu = D\,\partial_{xx}G_\mu/2 + \eta(\tilde{G}_\mu^2 - G_\mu) \ . \tag{2.8}$$

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For, in this limit the lhs yields $G_{\mu+d\mu} \to G_\mu + \partial_\mu G_\mu d\mu$, and on rhs the variance of v shrinks proportionally to $d\mu$, so that $\overline{G}_\mu \to G_\mu + [D + O(x)]\, d\mu\, \partial_{xx} G_\mu/2$. When the two sides are set equal, finite terms cancel out, and to the first order in $d\mu$ the above equation obtains near $x = 0$, which will be the relevant point.

It is easily perceived, and is discussed in detail in §3, that in the limit of no branching (i.e., $\eta \to 0$) eq. 2.8 reduces to a pure diffusion equation for $G_\mu(x)$, similar to the equation given by Bond et al. 1991 and recalled in the Introduction. The opposite limit of branching with no Gaussian noise yields, as we shall see in §5, the Smoluchowski equation for the kinetics of $N(M, t)$ under binary interactions discussed by CCM. We next substantiate these two limits by examining the relationship of $G_\mu(x)$ with $N(M, t)$, and that of the generation number μ with the resolution S or with the physical time t.

We stress that solving eq. 2.8 for $G_\mu(x)$ is *equivalent* to computing the single moments of Z directly from the relations 2.3 and then synthetizing $G_\mu(x)$ from its expansion 2.5. The advantage of following this latter route is that the recursion relations are very simple, and especially suited for numerical work. On the other hand, the master eq. 2.8 is convenient for making contact with previous work in limiting cases and for discussing the balance of the two competing modes. The statistical effects of this competition are illustrated in fig. 2.

3. FROM THE CAYLEY TREES TO THE MASS FUNCTION

We first note that in the limit of no branching (i.e., $\eta \to 0$) the remaining

terms of eq. 2.8 yield the structure of a diffusion equation

$$\partial_\mu G_\mu = D\,\partial_{xx}G_\mu/2 \ . \tag{3.1}$$

This, by the gauge (structure-preserving) transformation

$$d\mu = -\frac{1}{2}\,d\,ln\,S \ , \qquad dx^2 \propto d\delta^2\,D/S \ , \tag{3.2}$$

with the condition $D\,d\mu > 0$ (see eq. 2.4), can be made identical with the equation (Bond et al. 1991, Lacey & Cole 1993)

$$\partial_S Q = \partial_{\delta\delta}Q/2 \ , \tag{3.3}$$

which governs the evolution of $Q(\delta, S)\ d\delta$, the density of trajectories (random walks) of δ as the resolution $S \propto \sigma^2 \propto M^{-2a}$ is increased or the scale is decreased.

Because of its central role in what follows, we discuss the gauge 3.2 in more detail. We first integrate eq. 3.2b to

$$x \propto (\delta - \delta_c)/\sigma \ , \tag{3.4a}$$

so that $x = 0$ corresponds to $\delta = \delta_c$; thus the collapse threshold is embedded in the tree formalism as a zero point for the tree variable x.

Then we specify (see fig. 3 and its caption) the relationship of the remaining tree coordinate μ and of its initial value μ_o with the *physical* variables M and t. At each time t the resolution scale $S = M^{-2a}$ corresponding to a given generation number μ is reckoned from a minimum $S_o \propto [M_c(t)/\epsilon]^{-2a}$ with $\epsilon \ll 1$, corresponding to a maximum mass $M_{max} \equiv M_c(t)/\epsilon \gg M_c(t)$, so that eq. 3.2a is integrated

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in the form

$$\mu - \mu_o = -\frac{1}{2} \, ln \, \frac{S}{S_o} = ln \, (\epsilon \, m)^a \; . \tag{3.4b}$$

Note that S_o contains the t-dependence of the initial condition μ_o for each realization of the tree. The tree coordinates μ_o and $\mu - \mu_o$ are used as independent ones in what follows. The mass scale M is an independent variable in the frame of the *physical* coordinates (the plane M, t in fig. 3), but in terms of the *tree* coordinates becomes a function of m and t through the relation $M = m \, M_c(t)$ (see also caption to fig. 3).

Having so specified the relations between the physical and the tree variables, we now elaborate a procedure for computing the mass function from the tree. Following the papers by Bond et al. 1991 and Lacey & Cole 1993, the PS selection rule in terms of $Q(\delta, S)$ and $N(m) = N(M)dM/dm$ is written as

$$N(m) \, m \, dm = -\frac{\rho}{M_c} \, d \int_{-\infty}^{\delta_c} d\delta \, Q = -\frac{\rho}{M_c} \, dS \, [\partial_\delta Q/2]_{\delta_c} \; . \tag{3.5}$$

The second equality, which expresses the mass fraction as a density in resolution of flux across the (absorbing) boundary δ_c, is formally provided by integration over δ of eq. 3.3. We note that within the above theory a Laplace transformation relates $N(M)$ to $Q(\delta, S)$. In fact, the integral form of the above relation may be recovered by applying the operator $[\partial_\delta]_{\delta_c}$ to both sides of the integral relation

$$\int dm \, N(m) e^{-m(\delta - \delta_c)} = \frac{\rho}{M_c} \int dS \, Q/2 \; . \tag{3.6}$$

In other words, the selection rule and the diffusion equation imply that $\int dS \, Q/2$ is the Laplace transform of $N(m) \, M_c/\rho$. [2]

Based on the close similarity of eqs. 3.1 and 3.3 when the gauge eqs. 3.2 are considered, we propose the following general form of the selection rule:

$$\frac{N(m)}{N_T} \, m \, dm = -d \int dx \, G_\mu(x) = -d\mu \, [\partial_x G_\mu(x)]_o = -d\mu \, \langle Z_\mu \, e^{-Z_\mu} \rangle. \quad (3.7)$$

Here $N_T(t) \equiv \int N(m) \, dm \propto \rho/M_c$, and the derivative is computed at $x = 0$, corresponding to $\delta = \delta_c$ after the gauge transformation 3.4a; note from the second term that additive components of G independent of μ will not matter. In integral form, the fraction of condensed mass is

$$\int dm \, N(m) \, m = -N_T \int d\mu \, [\partial_x G_\mu]_o \, , \quad (3.8)$$

where the last integral is actually t-independent (see footnote [2]). But the above relation also obtains by differentiation ∂_x at $x = 0$ of the relation

$$\int dm \, N(m) \, e^{-m \, x} = N_T \int d\mu \, G_\mu(x) \, , \quad (3.9)$$

In the following, unless otherwise specified, the integrals are meant to run over the full range of the variables. These are as follows: $\delta \in [-\infty, +\infty]$, $m \in [M_{min}, M_c/\epsilon]$, and $\mu \in [-\infty, \mu_o]$. The integrals over μ ranging in the last interval may be rewritten as integrals over $\mu - \mu_o$ in the corresponding range, and are explicitly t-independent.

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which expresses the generating function, normalized to N_T, as a Laplace transform of the mass distribution. The latter then obtains by anti-transforming $G_\mu(x)$.

Equivalently, all successive moments of $N(m)$ are obtained by successive differentiation of eq. 3.9. For example, the zeroth-order moment is given by $\int dm N(m) = N_T \int d\mu G_\mu(0)$, which fixes the normalization $\int d\mu G_\mu(0) = 1$; the 1st moment is given by eq. 3.8, consistent with the differential form 3.7.

4. DISORDER: THE PS LIMIT

We now show how the PS mass distribution may be derived from eq. 3.7 in the limit of no branching, that is, $\eta \to 0$.

The latter equation involves

$$[\partial_x G_\mu]_o = \langle\, Z_\mu\, e^{-\,Z_\mu}\rangle = \sum_{k=1}(-1)^{k-1}\langle Z^k\rangle_\mu/(k-1)! \quad . \tag{4.1}$$

The single moments of Z will be *directly* derived from the recursion relation eq. 2.3a (with $D = 1$) in the form

$$\langle Z^k\rangle_{\mu+d\mu} = \int dv\, \frac{e^{-v^2/2d\mu}\,(1+v)^k}{\sqrt{2\pi d\mu}}\,\langle Z^k\rangle_\mu \quad . \tag{4.2}$$

Expanding the binomial around $v = 0$ to the lowest (second) significant order, integration of eq. 4.2 yields

$$\langle Z^k\rangle_{\mu+d\mu} = \left[1 + \frac{k^2 - k}{2}\, d\mu\right]\langle Z^k\rangle_\mu \quad . \tag{4.3}$$

In the continuum limit $d\mu \to 0$ this yields

$$d\, ln\langle Z^k\rangle_\mu = \frac{k^2 - k}{2}\, d\mu \quad , \tag{4.4}$$

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which using the gauge 3.2a (i.e., $d\mu = -d\ln\sigma$) integrates to

$$\langle Z^k \rangle_\mu / \langle Z^k \rangle_o = (\sigma_o/\sigma)^{\frac{k^2-k}{2}} . \tag{4.5}$$

This relation implies that the first moment of Z is a constant relative to μ, which is a natural consequence of the property $\overline{w} = 1$. The next two moments yield the leading contributions to the sum 4.1. In fact, following eq. 4.5 and keeping only the μ-dependent terms (as noted just after eq. 3.7), we find

$$\langle\, Z_\mu\, e^{-\,Z_\mu} \rangle = \langle Z \left[1 - Z + \frac{Z^2}{2} + ... \right] \rangle = Z_o^2 \,\frac{\sigma_o}{\sigma} \left[1 - \frac{Z_o}{2}\,\frac{\sigma_o^2}{\sigma^2} + O\!\left(\frac{\sigma_o^5}{\sigma^5}\right) \right] . \tag{4.6}$$

Recalling from §3 that $\sigma_o/\sigma = [M/M_{max}]^a = (\epsilon\, m)^a$, it is seen that $\sigma_o/\sigma < 1$ holds for $M < M_{max} = M_c/\epsilon$, and the last equality can be resummed to within $O(\epsilon^{5a})$ to yield

$$[\partial_x G_\mu]_o \approx Z_o^2 \,\frac{\sigma_o}{\sigma}\, e^{-Z_o\, \sigma_o^2/2\,\sigma^2} = Z_o^2\, (\epsilon\, m)^a\, e^{-Z_o(\epsilon\, m)^{2a}} . \tag{4.7}$$

The tree by itself, as any statistics, does not specify the dynamics of gravitational collapses, and provides only scaling behaviors: these, on substituting eq. 4.7 in eq. 3.7, are

$$N(m) = \frac{\rho}{M_c}\, m^{-2}\, \frac{d\ln\sigma}{d\ln m}\, [\partial_x G_\mu]_o \approx \frac{2\,a\,\rho\, Z_o^2}{M_c}\, (\epsilon\, m)^{-2+a}\, e^{-Z_o\, (\epsilon\, m)^{2a}/2} , \tag{4.8}$$

and turn out to be the same as in the PS distribution. As to the constants which do carry dynamical information, the initial condition Z_o may be set to δ_c, the counterpart here of the boundary condition used by Bond et al. 1991. In addition, the rescaling $N(\epsilon\, m) = N(m)/\epsilon$ holds, and the overall normalization follows from

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the requirement $\int d\mu \, G_\mu(0) = 1$ derived in §3. Thus the full PS expression 1.1 is recovered.

5. BRANCHING: THE SMOLUCHOWSKI LIMIT

We now consider eq. 2.8 in the opposite limit of branching with no Gaussian noise. Then the remaining terms on rhs read

$$\partial_\mu G_\mu = \eta(\tilde{G}_\mu^2 - G_\mu). \tag{5.1}$$

Equivalently, the change of single moments of Z may be derived *directly* from the recursion relation 2.3b with no noise, to yield

$$\langle Z^m \rangle_{\mu+d\mu} = d\mu \int \int dZ_1 \, dZ_2 \, P(Z_1) \, P(Z_2) \, (Z_{1\mu} + Z_{2\mu})^m$$

$$+(1 - d\mu) \int dZ \, P(Z) \, Z_\mu^m \; ; \tag{5.2}$$

here the coupling parameter η has been absorbed into a rescaling of μ for convenience as will become apparent after eq. 5.11. Expanding the binomial, this becomes

$$\langle Z^m \rangle_{\mu+d\mu} = d\mu \int \int dZ_1 d\, Z_2 \, P(Z_1) \, P(Z_2) \sum_{k=1}^{m} \frac{m!}{k! \, (m-k)!} \, Z_{1\mu}^k \, Z_{2\mu}^{m-k}$$

$$+(1 - d\mu)\langle Z^m \rangle_\mu. \tag{5.3}$$

In the continuous limit $d\mu \to 0$ this yields

$$\partial_\mu \frac{\langle Z^m \rangle_\mu}{m!} = \sum_{k=1}^{m} \frac{\langle Z^k \rangle_\mu}{k!} \frac{\langle Z^{m-k} \rangle_\mu}{(m-k)!} - \frac{\langle Z^m \rangle_\mu}{m!} \; . \tag{5.4}$$

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This relation has a structure similar to the Smoluchowski equation which governs the kinetics of binary aggregations of definite condensations (CCM):

$$\frac{\partial N}{\partial t} = \frac{1}{2} \int_0^M dM' \; K(M', M - M', t) \; N(M', t) \; N(M - M', t)$$

$$- N(M, t) \int_0^\infty dM' \; K(M, M', t) N(M', t) \; . \tag{5.5}$$

The kernel K represents the rate of aggregation in a system of condensations with relative velocity V and interaction cross section Σ, and averages to $\tau^{-1} = N_T \overline{\Sigma V}$ in terms of the component number density $N_T(t)$.

Indeed, we show next that the above equation may be identified with eq. 5.4 (or with the equivalent eq. 5.1) when dealing with high-contrast condensations insensitive to the perturbation field; that is, in the limit of no disorder (with $\delta = \sigma$) and of high-contrast condensations (i.e., $\delta_c \to 0$). For a constant kernel K we again proceed directly.

To this end, we first rewrite the discretized form of eq. 5.5, after dividing by $N_T/2$, in the form

$$\frac{2}{N_T^2} \frac{\partial N_M}{\partial t} + \frac{N_M}{N_T} = \sum_{M'} \frac{N_{M-M'}}{N_T} \frac{N_{M'}}{N_T} - \frac{N_M}{N_T} \; , \tag{5.6}$$

where the parameter K has been absorbed into a rescaling of t for convenience which again will become apparent after eq. 5.11 below. We then express the t-derivative in terms of the tree variable μ. The gauge eq. 3.2a yields $d\mu = -\frac{1}{2} d \ln S = a \, d \ln M$. In the tree frame (as we discussed in §3) $M = m \, M_c(t) \propto m/N_T$ hold, and one obtains

$$d\mu \propto -\frac{dN_T}{N_T} = \frac{1}{2} N_T \, dt \; , \tag{5.7}$$

with the last equality coming from eq. 5.5 integrated over M. Then eq. 5.6 can be written as

$$\frac{\partial}{\partial \mu} \frac{N_M}{N_T} = \sum_{M'} \frac{N_{M-M'}}{N_T} \frac{N_{M'}}{N_T} - \frac{N_M}{N_T} \ . \tag{5.8}$$

Now the above equation exhibits even more clearly a structure similar to eq. 5.4, and is identical to it provided that

$$\frac{N_M}{N_T} \propto \frac{\langle Z^m \rangle_\mu}{m!} \tag{5.9}$$

holds. The simplest way to prove this relationship is to to write the integral relation 3.9 with the shorthand $1 + x = w$ in the form

$$\int \frac{N(m)}{N_T} w^m \, dm = \int d\mu \langle e^{-w Z_\mu} \rangle = \sum_k w^k \int d\mu \frac{\langle Z^k \rangle_\mu}{k!} \ , \tag{5.10}$$

with the signs $(-1)^k$ included into $\langle Z^k \rangle$ in view of the invariance of eq. 5.4 relative to such transformations. Then we note that eq. 5.4 for $m = 1$ implies $d \ln \langle Z \rangle_\mu \propto d\mu$, and hence $\langle Z \rangle_\mu \propto N_T$ by eq. 5.7. For $m = 2$ it implies $\langle Z^2 \rangle \propto N_T$ asymptotically when $N_T \ll N_{To}$ holds; similarly for the higher orders. With $\langle Z^k \rangle_\mu / N_T \, k! \to$ const relative to μ, the integration on rhs of eq. 5.10 reduces to $\int d\mu N_T = \int d\mu e^{-\mu}$ in view of eq. 5.7, and yields a constant factor. Thus, writing the integral on the lhs of eq. 5.10 in discrete form, we find

$$\sum_m N_m w^m \propto \sum_k w^k \langle Z^k \rangle_\mu / k! \ , \tag{5.11}$$

where of course the dummy indexes k and m may be identified. We now apply to both members the operator $[\partial_w^m]_{w=0}$, which in fact is appropriate for high-contrast structures with $\delta_c \to 0$ and $\delta \approx \sigma$. The result is the relation 5.9, thus

completing the identification of the Smoluchowski equation of the form 5.5 with the tree recursion relations of the form 5.4.

The proportionality constant in eq. 5.11 translates into a rescaling by a constant factor of the correspondence between the independent variable in eq. 5.4 (which in full is $\mu\eta$), and that in eq. 5.6 (which in full is t/τ). The statement that $\sum_m \langle Z^m \rangle_\mu / N_T\, m!$ becomes time-independent holds for the solutions $N(M, t)$ of the Smoluchowski equation in their asymptotic, self-similar stage where $N(M)/N_T^2$ is time-independent.

The equivalence of the Smoluchowski equation 5.5 with a Cayley tree can also be proved when the kernel $K(M, M')$ is mass-dependent. Then the equivalence may be recovered by going through the Laplace transform of eq. 5.5, as has been carried out explicitly by Peshanski (1992) for the case of multiplicative kernels $K(M, M') = K(M)\,K(M')$. The link between kernel and weight distribution on the tree $r(u)$ is explicitly given by

$$K(m) = C \int_0^\infty du\, r(u)\, e^{um} \, , \tag{5.12}$$

which amounts again to a Laplace transformation. The constant $C \propto \tau^{1/2}$ contains the normalization of K, since $r(u)$ is normalized to 1.

6. DISORDER AND BRANCHING SUPERPOSED

We now compute numerically the mass function in conditions where both disorder and branching are effective. As has been said, for numerical work it is much easier and faster to use *directly* the recursion relations (eq. 2.3) for the

partition function Z_μ – rather than the master equation 2.8.

This is done by the following procedure. At each step μ of the tree a numerical algorithm extracts the weights v from a Gaussian distribution, and uses the first recursion relation 2.3 with probability $1 - \eta\,\Delta\mu$, or the second with probability $\eta\,\Delta\mu$, to construct the partition function at the next step. In the present computations we use a delta-function $r(u) = \delta(u)$, corresponding after eq. 5.12 to a constant interaction kernel. In fig. 2 we have shown the resulting trajectories of the contrast for increasing S, which are related to the tree variables x and μ by the gauge eqs. 3.4.

For each tree, we generate the entire Z_μ and its moments. The procedure is repeated for a large number of trees (up to 10^3). The moments of the partition function are used to compute

$$[\partial_x G_\mu]_o = \sum_{k=1}(-1)^{k-1}\langle Z_\mu^k\rangle/(k-1)!, \tag{6.1}$$

where the sum we actually used runs up to $k = 10$. Then at any given t we compute the mass distribution $N(M, t)$ following eq. 3.7, and check that the total mass is conserved in time to within 1% when 10^3 trees are used.

The result depends on the branching probability η, which acts like an effective coupling constant for binary interactions. For small $\eta \lesssim 10^{-2}$ the diffusion part of eq. 2.8 dominates and one recovers the PS mass distribution. For increasing values of η the resulting distributions become steeper, while the cutoff at large masses tends to straighten up. When $\eta \sim 10^{-1}$ the distribution reaches its *asymptotic* form, a pure power law with logarithmic slope close to -2; see fig. 4

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There are two reasons why even a small value of η is effective. First, note that on transforming eq. 2.8 following the gauge eqs. 3.4 the effective coupling parameter is $\eta/S \propto \eta m^{2a}$, that is, a coupling more effective at larger masses. Second, a kernel constant in time, as considered here, maximizes the interactions because it implies no dependence of the number density on ambient evolution. This is physically realistic for interactions within large scale structures surviving for longer then τ.

These results compare interestingly with the results from cosmological N-body simulations with a large dynamical range and highly resolved data analysis, like those performed by Brainerd & Villumsen 1992 and Jain & Bertschinger 1994. These papers agree in finding a slower evolution and an excess at large masses compared with the PS mass distribution, with the former finding consistency with a simple power law distribution, approximatively M^{-2} within the walls and filaments. These features are consistent with our findings.

In terms of overdensities vs. resolution as used in fig. 2, the interactions cause not only an obvious density decrement of the trajectory distribution toward larger coalesced masses, but also a skewness relative to the Gaussian counterpart.

7. DISCUSSION

The existing theories for the shape and the evolution of the mass distribution $N(M,t)$ fail to capture the full complexity of cosmic structures. The PS theory, with its recent elaborations, provides with eq. 1.1 the best quantification for

the scenario envisaging hierarchical collapses from initial density perturbations. Yet the result differs as to amplitude, shape, and evolution from data or from simulations and from realistic excursion set computations with the same input parameters. The alternative mode to hierarchically building up structure is based on binary aggregations between high-contrast condensations. This predicts non-Gaussian formation of rare large objects, and describes more closely the erasure of substructures within a structure by resolving timescales different from its dynamical time t_d. But it requires input, at least initially, of formed condensations into an environment protected from the Hubble expansion (CCM).

We submit that both these modes constitute only partial representations of the evolution of the density field. They select either purely Gaussian random walks of the linear density contrasts as functions of the resolution, up to crossing the threshold of nonlinearity; or trajectories "branching" stochastically only at large values of the contrast. Correspondingly, the PS function 1.1 and the Smoluchowski aggregation equation 5.5 correspond to *restricted*, "macroscopic" averages from a *complete* " microscopic" field statistics which treats aggregations and direct collapses as proceeding together at similar contrast levels.

We propose such a complete statistics which combines, in the form of a Cayley tree (or random cascade) as represented in fig. 1, random walk and stochastic branching of fluctuations into one partition function governed by the single *master* equation 2.8. We also propose the Laplace transform relationship eq. 3.8 between the tree generating function $G_\mu(x)$ and the mass distribution $N(M, t)$.

We have proved that our proposals indeed yield in the appropriate limits both the diffusion equation (§4) with the associated PS function eq. 1.1, and the Smoluchowski equation (§5). The former limit applies to Gaussian fluctuations that *independently* reach the nominal threshold for collapse and virialization. The latter limit is derived with no formal recourse to a threshold, and describes binary *interactions* of an initially given set of already formed condensations. Pleasingly, from averaging over the tree in the latter limit one obtains a mean-field, kinetic *equation* with different solutions depending on the kernel, and evolving away from the initial form; in the former limit, one instead obtains a single PS *function*.

Both limiting processes are hierarchical and imply "merging", i.e., inclusion of smaller condensations into larger ones. But pure DHC in fact envisages only reshuffling of (generally) many condensations which belong to lower hierarchical levels into a higher level on a *larger* mass scale at a *subsequent* time, strictly following the conditions set ab initio in the linear fluctuation field. The natural quantification of this process is provided by the conditional probability that a trajectory up-crossing the threshold at the resolution S_2 had a previous up-crossing at $S_1 > S_2$ (see Bond et al 1991; Bower 1991; Lacey & Cole 1993).

Pure aggregation, on the other hand, envisages pairs of condensations *coalescing* into a third, at a *similar* – and high – contrast level. Here the nonlinear interactions are stochastically set by ambient density and relative velocities, and may be triggered at any time when larger scale structures outline volumes with expansion slowed down relative to the general "field".

The Cayley tree approach not only proves successful in deriving the two, oppositely extreme modes for hierarchically building up structures, but also provides a number of links between them: the common tree algorithm or the master equation 2.8; the Laplace transform relationship equation 3.8 of $G_\mu(x)$ with $N(M,t)$, which extends the PS selection rule to interacting fluctuations; and the tree variable μ that includes both the resolution S (used to derive the PS function) and the physical time t (used to derive the Smoluchowski equation), as stressed below.

In fact, we have seen that $d\mu = -\frac{1}{2} d\, lnS$ holds, appropriate to the former case. On the other hand, from the relation $M = m\, M_c(t) \propto m/N_T$ one obtains $\partial_t \mu \propto -d\, lnN_T/dt$, which is the eq. 5.7 used in deriving the Smoluchowski equation. The constant factors (and specifically the spectral index n, the last remnant of the initial perturbation spectrum) are actually irrelevant in the subsequent identification of eq. 5.5 with the tree recursion relation in the absence of Gaussian noise, as expected.

Above all, the tree formalism describes with equal ease the mixing and competition of the two pure modes; that is, it describes condensations interacting and aggregating while still growing or collapsing. The process may be computed either from the master equation, or directly in terms of the tree partition function, which is very convenient and fast on computers. These computations yield, when branching is fully effective, $N(M,t)$ in the form of a steep, slowly lowering power law with logarithmic slope ≈ -2. The result is due to the continuous input from the Gaussian noise over many scales, combined with the increasing effect of the

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branching mode at larger masses.

In fact, even a value of $\eta < 1$ is effective because the relevant coupling parameter $\eta/S \propto \eta \, m^{2a}$ increases at higher masses. Actually, the interactions are maximized by a kernel constant in time as considered here, corresponding to densities and velocities unaffected by expansion. This is physically realistic for interactions taking place within large-scale structures which survive for times longer that τ. In these conditions high-mass condensations form faster than the *Gaussian* rate within large scale structures, not unlike the formation action seen in large N-body simulations (Brainerd & Villumsen 1992; Babul & Katz 1992). Observations show that groups are concentrated in filaments and sheets outlined by redshift surveys of galaxies (Ramella et al. 1990) and that the mass distribution over scales from 10^{13} $M_\odot$ to several 10^{15} $M_\odot$ is consistent with a steep power law (Giuricin et al. 1993).

We shall discuss elsewhere the relationship of our approach with the dynamical descriptions of matter field under gravity – e.g., the adhesion model (see Shandarin & Zel'dovich 1989) or the frozen-flow approximation (Matarrese et al. 1992) – which include shear effects in addition to interactions of comparable fluctuations. Conditions of aggregating interactions protected from the Hubble expansion within larger structures are likely to underlie apparently diverse astrophysical phenomena, from formation of one large cD-like galaxy by strong interactions of members in groups, to the very formation and growth of groups and clusters within large scale filaments and sheets. Toward such a complex emergence of cosmic structure Cayley

trees provide a unifying approach.

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Fig. 1. A schematic representation of a Cayley tree. The random process is represented as a cascade proceeding from the bottom up. At each step (labeled by the generation number $\mu + d\mu$) a random weight $w = 1 + v$ is extracted according to a Gaussian distribution $g(v)$. In addition, the path may either result from coalescing (with probability $\eta \, d\mu$) of two branches that were still distinct at μ, or proceed along a single branch. The partition function at a point $\mu + d\mu$ is constructed by summing the products of the weights over all paths leading to it. The heavy line marks a specific path on the tree with weights w_1, w_2, ... w_7.

Fig. 2. Contrast vs. resolution (in arbitrary units) of a number of trajectories computed from numerical realizations of the tree (details are given in §6). *Top panel*: Pure Gaussian noise resulting in random walks with dispersion increasing with increasing resolution (or decreasing mass). *Bottom panel*: Effects of including random branching (with $\eta = 0.1$) in the statistics of the trajectories; the branching points are marked by a dot. The study of the apparent differences between the two panels constitutes the thrust of this paper.

Fig. 3. Relations between the physical variables M and t and the tree coordinates μ and μ_o . At a given t (*top panel*), the initial value S_o for the resolution corresponds to a maximum mass scale $M_{max} = M_c(t)/\epsilon \gg M_c(t)$. It also corresponds

to a value of the tree generation number $\mu_o \propto lnM_{max}$ following eq. 3.2. So

$\mu - \mu_o \propto \ln m$ holds, with $m \propto M/M_c(t)$. For a given mass M (*bottom panel*), dif-

ferent values of t correspond to various starting points μ_o, and to different values

of μ visualized by the intercepts of a horizontal line for increasing t.

Fig. 4. Mass function $N(M)$ resulting from numerical realizations of trees with

both branching and disorder. The partition function and the related moments are

computed from the eq. 2.3. *Dotted curve*: $\eta = 0$; *dashed curve*: $\eta = 0.05$; *solid*

curve: $\eta = 0.1$. The first is identical with the Press & Schechter function, eq. 1.1.